

# Degenerate first-order Hamiltonian operators of hydrodynamic type in 2D

Andrea Savoldi

Department of Mathematical Sciences, Loughborough University  
Leicestershire LE11 3TU, Loughborough, United Kingdom  
e-mail:

A.Savoldi@lboro.ac.uk

## Abstract

First-order Hamiltonian operators of hydrodynamic type were introduced by Drubrovin and Novikov in 1983. In 2D, they are generated by a pair of contravariant metrics  $g, \tilde{g}$  and a pair of differential-geometric objects  $b, \tilde{b}$ . If the determinant of the pencil  $g + \lambda \tilde{g}$  vanishes for all  $\lambda$ , the operator is called degenerate. In this paper we provide a complete classification of degenerate two- and three-component Hamiltonian operators. Moreover, we study the integrability, by the method of hydrodynamic reductions, of 2+1 Hamiltonian systems arising from the structures we classified.

MSC: 37K05, 37K10, 37K25.

Keywords: Degenerate Metric, Hamiltonian Operator, Hamiltonian System, Hydrodynamic Reductions.

## 1 Introduction

The theory of first-order Hamiltonian operator of differential-geometric type has been developed in the last three decades by several authors, starting from the pioneering work of Dubrovin and Novikov [4]. In one-dimensional case, these structures are given by

$$P^{ij} = g^{ij}(\mathbf{u}) \frac{d}{dx} + b_k^{ij}(\mathbf{u}) u_x^k, \quad (1)$$

where  $\mathbf{u} = (u^1, \dots, u^n)$  are local coordinates depending on  $x$ ,  $i, j, k = 1, \dots, n$ , and  $u_x^k = \frac{du^k}{dx}$ . Dubrovin and Novikov proved that in the non-degenerate situation, namely  $\det(g^{ij}) \neq 0$ , (1) defines a Poisson bracket through

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j},$$

if and only if  $g^{ij}$  is a flat pseudo-Riemannian metric and the coefficients  $\Gamma_{jk}^i = -g_{jm} b_k^{mi}$  are the Levi-Civita connection of the metric  $g_{ij}$  (where  $g^{im} g_{mj} = \delta_j^i$ ). Thus, in flat coordinates, any non-degenerate Hamiltonian operator (1) assumes constant form. In the case where the metric  $g$  is degenerate, that is,  $\det(g^{ij}) = 0$ , this result does not hold. Grinberg [12] and later Bogoyavlenskij

[1, 2] firstly investigated this case, and recently we provided a complete list of two- and three-component Poisson structures with degenerate metric [16].

First-order Hamiltonian operators of differential-geometric type naturally arise in the study of quasilinear systems (systems of hydrodynamic type). In 1+1 dimensions they are given by

$$u_t^i + V_j^i(\mathbf{u})u_x^j = 0.$$

Such systems are called Hamiltonian if they can be written in the form

$$u_t^i + P^{ij}\delta_j h = 0,$$

where  $\delta_j = \delta/\delta u^j$  is the variational derivative,  $h = h(\mathbf{u})$  is the Hamiltonian density, and  $P^{ij}$  is a Hamiltonian operator of hydrodynamic type (1). It was conjectured by Novikov that a combination of the Hamiltonian property with the diagonalizability of the matrix  $V_j^i$  implies the integrability. This conjecture was proved by Tsarev in [20], who established the linearizability of such systems by the generalised hodograph transform.

A generalisation of hydrodynamic type systems in 2+1 dimensions is given by

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0, \quad (2)$$

where  $\mathbf{u} = (u^1, \dots, u^n)$ ,  $u^i = u^i(t, x, y)$  and  $A, B$  are  $n \times n$  matrices. Systems of this type describe many physical phenomena. In particular, important examples occur in gas dynamics, shallow water theory, combustion theory, nonlinear elasticity, magneto-fluid dynamics, etc. A system (2) is called Hamiltonian if it can be written in the form  $u_t^i + P^{ij}\delta_j h = 0$ , where  $P^{ij}$  is a 2D first-order Hamiltonian operator of differential-geometric type, namely

$$P^{ij} = g^{ij}(\mathbf{u})\frac{d}{dx} + b_k^{ij}(\mathbf{u})u_x^k + \tilde{g}^{ij}(\mathbf{u})\frac{d}{dy} + \tilde{b}_k^{ij}(\mathbf{u})u_y^k. \quad (3)$$

In 2+1 dimensions, a quasilinear system is said to be integrable if it can be decoupled in infinitely many ways into a pair of compatible  $m$ -component one-dimensional systems in Riemann invariants [6]. Ferapontov and Khusnutdinova proved that the requirement of the existence of sufficiently many  $m$ -component reductions provides an effective classification criterion. This method of hydrodynamic reductions, which is a natural analogue of the generalised hodograph transform in higher dimensions, leads to finite-dimensional moduli spaces of integrable Hamiltonians.

The purpose of this paper is two-fold. Starting from the classification of degenerate brackets in 1D, we want to describe degenerate Hamiltonian operators of hydrodynamic type in 2D, that is, operators of the form (3) such that  $\det(g + \lambda \tilde{g}) = 0$  holds  $\forall \lambda$  (precise definition follows). Our analysis leads to a complete classification of two- and three-component degenerate structures (Section 2). Secondly, we study the integrability, by the method of hydrodynamic reductions, of Hamiltonian systems arising from three-component structures we classified (Section 3).

## 2 Degenerate Hamiltonian operators in 2D

The problem of classification of multidimensional Hamiltonian operators was proposed by Dubrovin and Novikov in [5], and thoroughly investigated by Mokhov [13, 14]. Some results in the classification of 2D non-degenerate Hamiltonian operators were recently obtained in our paper [9].

A first-order multidimensional Hamiltonian operator of differential-geometric type (Dubrovin-Novikov type) is defined by

$$P^{ij} = \sum_{\alpha=1}^d g^{ij\alpha}(\mathbf{u})\frac{d}{dx^\alpha} + b_k^{ij\alpha}(\mathbf{u})u_{x^\alpha}^k, \quad (4)$$

where  $\mathbf{u} = (u^1, \dots, u^n)$  are local coordinates on a certain smooth  $n$ -dimensional manifold  $M$  or a domain of  $\mathbb{R}^n$ , and  $\mathbf{x} = (x^1, \dots, x^d)$  are independent variables.

As in the one-dimensional case, the condition of skew-symmetry and the Jacobi identity for a Hamiltonian operator of the form (4) impose very severe restrictions on the coefficients  $g^{ij\alpha}(\mathbf{u})$  and  $b_k^{ij\alpha}(\mathbf{u})$ . In particular, Mokhov proved the following general statement:

**Theorem 1** ([15]). *Operator of the form (4) is a Hamiltonian operator, i.e. it is skew-symmetric and satisfies the Jacobi identity, if and only if the following relations for the coefficients of the operator are fulfilled:*

$$g^{ij\alpha} = g^{ji\alpha}, \quad (5a)$$

$$\frac{\partial g^{ij\alpha}}{\partial u^k} = b_k^{ij\alpha} + b_k^{ji\alpha}, \quad (5b)$$

$$\sum_{(\alpha, \beta)} (g^{si\alpha} b_s^{jr\beta} - g^{sj\beta} b_s^{ir\alpha}) = 0, \quad (5c)$$

$$\sum_{(i, j, r)} (g^{si\alpha} b_s^{jr\beta} - g^{sj\beta} b_s^{ir\alpha}) = 0, \quad (5d)$$

$$\sum_{(\alpha, \beta)} \left[ g^{si\alpha} \left( \frac{\partial b_s^{jr\beta}}{\partial u^q} - \frac{\partial b_q^{jr\beta}}{\partial u^s} \right) + b_s^{ij\alpha} b_q^{sr\beta} - b_s^{ir\alpha} b_q^{sj\beta} \right] = 0, \quad (5e)$$

$$g^{si\beta} \frac{\partial b_q^{jr\alpha}}{\partial u^s} - b_s^{ij\beta} b_q^{sr\alpha} - b_s^{ir\beta} b_q^{js\alpha} = g^{sj\alpha} \frac{\partial b_q^{ir\beta}}{\partial u^s} - b_s^{ji\alpha} b_q^{sr\beta} - b_q^{is\beta} b_s^{jr\alpha}, \quad (5f)$$

$$\begin{aligned} & \frac{\partial}{\partial u^k} \left[ g^{si\alpha} \left( \frac{\partial b_s^{jr\beta}}{\partial u^q} - \frac{\partial b_q^{jr\beta}}{\partial u^s} \right) + b_s^{ij\alpha} b_q^{sr\beta} - b_s^{ir\alpha} b_q^{sj\beta} \right] + \sum_{(i, j, r)} \left[ b_q^{si\beta} \left( \frac{\partial b_k^{jr\alpha}}{\partial u^s} - \frac{\partial b_s^{jr\alpha}}{\partial u^k} \right) \right] \\ & + \frac{\partial}{\partial u^q} \left[ g^{si\beta} \left( \frac{\partial b_s^{jr\alpha}}{\partial u^k} - \frac{\partial b_k^{jr\alpha}}{\partial u^s} \right) + b_s^{ij\beta} b_k^{sr\alpha} - b_s^{ir\beta} b_k^{sj\alpha} \right] + \sum_{(i, j, r)} \left[ b_k^{si\alpha} \left( \frac{\partial b_q^{jr\beta}}{\partial u^s} - \frac{\partial b_s^{jr\beta}}{\partial u^q} \right) \right] = 0. \end{aligned} \quad (5g)$$

Relations (5a) and (5b) are equivalent to the skew-symmetry of the bivector (4), and relations (5c)–(5g) are equivalent to the fulfilment of the Jacobi identity for a skew-symmetric bivector of the form (4). The signs  $\sum_{(\alpha, \beta)}$  and  $\sum_{(i, j, k)}$  mean cyclic summation on the indicated indices. Notice that for  $n = 1$ , these conditions reduce to Grinberg's conditions [12].

In the one-dimensional case, Hamiltonian operator (1) is called *degenerate* if  $\det(g^{ij}) = 0$ . In the multidimensional situation, we have the following

**Definition 1.** *A  $d$ -dimensional operator of the form (4) is said to be degenerate if  $\det \left( \sum_{\alpha=1}^d \lambda_\alpha g^\alpha \right) = 0$  for any choice of  $\lambda_1, \dots, \lambda_d$ , i.e. if there is no linear combination of the metrics  $g^\alpha$  such that the determinant of this linear combination is non-zero.*

Let us point out that Theorem 1 does not assume non-degeneracy of operators or additional conditions on the coefficients of (4).

From Mokhov's conditions it immediately follows that each multidimensional Hamiltonian operator of the form (4) is always the sum of one-dimensional Hamiltonian operators with respect to each of the independent variables  $x^\alpha$ , [15].

Based on this result, and on the classification of one-dimensional degenerate Poisson structures of hydrodynamic type, we give a complete description of two- and three-component degenerate Hamiltonian operators for  $d = 2$ , namely

$$P^{ij} = g^{ij}(\mathbf{u}) \frac{d}{dx} + b_k^{ij}(\mathbf{u}) u_x^k + \tilde{g}^{ij}(\mathbf{u}) \frac{d}{dy} + \tilde{b}_k^{ij}(\mathbf{u}) u_y^k. \quad (6)$$

For simplicity, let us label the  $x$ -part and the  $y$ -part of the Hamiltonian operator (6) respectively with  $P_{(x)}$  and  $P_{(y)}$ .

Hamiltonian structure of the form (6) is called *trivial* if it is identically zero, or if it can be reduced to the form

$$\tilde{g}^{ij} = \xi g^{ij}, \quad \tilde{b}_k^{ij} = \xi b_k^{ij}, \quad (7)$$

for  $\xi$  constant. Notice that allowing linear change of the independent variables  $x, y$ , an operator satisfying (7) is essentially 1D.

**Remark.** Let us remark that if a pair of Hamiltonian operators defines a 2D structure, by (5) it easily follows that these two operators are compatible, and therefore they define a bi-Hamiltonian structure [14, 15]. Degenerate bi-Hamiltonian structures of hydrodynamic type were firstly investigated by Strachan [18, 19], revealing a nice relation with the theory of the analogous of Frobenius manifolds with degenerate metric.

## 2.1 Classification

The analysis of Mokhov's conditions (5) is not straightforward. In order to study two- and three-component structures, we fix the pair  $(g, b)$  given by the classification of 1D Hamiltonian operators [16]. This classification can be summarised in the following two theorems.

**Theorem 2.** *Any degenerate two-component Hamiltonian operator of Dubrovin-Novikov type in 1D can be brought, by a change of the dependent variables, to one of the following two canonical forms:*

$$P = \begin{pmatrix} d_x & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x & -\frac{u_x^2}{u^1} \\ \frac{u_x^2}{u^1} & 0 \end{pmatrix}. \quad (8)$$

**Theorem 3.** *Any degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 1D can be brought, by a change of the dependent variables, to one of the following canonical forms:*

- $\text{rank}(g) = 0$ :

$$P = \begin{pmatrix} 0 & u_x^3 & 0 \\ -u_x^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (9)$$

- $\text{rank}(g) = 1$ :

$$P = \begin{pmatrix} d_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x & u_x^3 & 0 \\ -u_x^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x & 0 & -\frac{u_x^3}{u^1} \\ 0 & 0 & 0 \\ \frac{u_x^3}{u^1} & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x & -\frac{u_x^2}{u^1} & -\frac{u_x^3}{u^1} \\ \frac{u_x^2}{u^1} & 0 & 0 \\ \frac{u_x^3}{u^1} & 0 & 0 \end{pmatrix}, \quad (10)$$

- $\text{rank}(g)=2$ :

$$P = \begin{pmatrix} 0 & d_x & 0 \\ d_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x & -\frac{u_x^3}{u^2} \\ d_x & 0 & 0 \\ \frac{u_x^3}{u^2} & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x & \frac{u_x^3}{u^3 u^1 - u^2} \\ d_x & 0 & \frac{-u_x^3}{u^3 u^1 - u^2} \\ \frac{-u_x^3}{u^3 u^1 - u^2} & \frac{u^3 u_x^3}{u^3 u^1 - u^2} & 0 \end{pmatrix}. \quad (11)$$

Once we have fixed the pair  $(g, b)$ , solving (5) we are able to find the pair  $(\tilde{g}, \tilde{b})$ . At this point, we look for canonical forms of 2D structures using transformations which preserve the form of the

first structure given by  $(g, b)$ . As we will see, in some cases these transformations are not enough to eliminate all the functional parameters appearing in the 2D structure.

Let us agree on some notation: if a function depends only on one variable, we denote with  $'$  the derivative with respect to that variable. Otherwise, if a function depends on more than one variable, say,  $f = f(u^1, \dots, u^n)$ , then we use  $\partial_i f = \frac{\partial f}{\partial u^i}$ . In Section 3, for simplicity, the derivative with respect to  $u^i$  will be denoted as  $f_{u^i}$ .

### 2.1.1 Two-component case

Here we provide a full description of the two-component case.

**Theorem 4.** *Any non-trivial degenerate two-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables, to the following form*

$$P = \begin{pmatrix} d_x + u^2 d_y + \frac{1}{2} u_y^2 & -\epsilon \frac{u_x^2 + u^2 u_y^2}{u^1} \\ \epsilon \frac{u_x^2 + u^2 u_y^2}{u^1} & 0 \end{pmatrix}, \quad (12)$$

where  $\epsilon$  can be either 0 or 1.

**Proof:**

First of all, the case  $g = \tilde{g} = 0$  gives no non-trivial solutions. In the case where the rank of the pencil  $g^{ij} + \lambda \tilde{g}^{ij}$  is constantly equal to one, there exists a coordinates system  $(u^1, u^2)$  where

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix},$$

here  $f = f(u^1, u^2)$  is some function. Let us fix the  $P_{(x)}$  structure.

**Case (8)<sub>1</sub>.** If  $b_k^{ij}$  are all identically zero, conditions (5) imply

$$f = f(u^2), \quad \tilde{b}_2^{11} = \frac{f'}{2},$$

and all other  $\tilde{b}_k^{ij}$  equal to zero. If  $f = \xi$  is constant, then  $\tilde{g} = \xi g$ . Otherwise, using a transformation which preserves  $P_{(x)}$ , that is, a suitable change of coordinates of the form  $u^1 = v^1 + \varphi^1(v^2)$ ,  $u^2 = \varphi^2(v^2)$ , we can easily reduce  $f$  to  $v^2$ , obtaining (12) with  $\epsilon = 0$ .

**Case (8)<sub>2</sub>.** Suppose  $b_2^{21} = -b_2^{12} = \frac{1}{u^1}$ . Conditions (5) imply

$$f = f(u^2), \quad \tilde{b}_2^{11} = \frac{f'}{2}, \quad \tilde{b}_2^{21} = -\tilde{b}_2^{12} = \frac{f}{u^1}.$$

If  $f = \xi$  is constant, then  $\tilde{g} = \xi g$ ,  $\tilde{b} = \xi b$ . Otherwise, let us assume  $f$  non-constant. Transformations which preserve  $P_{(x)}$  are given by  $u^1 = v^1$ ,  $u^2 = \varphi(v^2)$ , then we can always choose  $\varphi$  such that  $f$  reduces to  $v^2$  in the new coordinate system, obtaining (12) with  $\epsilon = 1$ . ■

### 2.1.2 Three-component case

The analysis of the three-component situation is more complicated. Let us consider separately the cases with respect to the rank of the pencil  $g_\lambda = g - \lambda \tilde{g}$ . We point out that in some cases the group of transformations preserving the first structure  $P_{(x)}$  is not sufficient to reduce the second structure  $P_{(y)}$  to something simpler. Thus, we will just consider the more general structure given by the solution of Mokhov's conditions. The results can be stated as follows.

**Theorem 5.**  $\text{Rank}(g_\lambda) = 0$ . Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables, to one of the following forms:

$$P = \begin{pmatrix} 0 & u_x^3 + u^1 u_y^3 & 0 \\ -u_x^3 - u^1 u_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u_x^3 + u^3 u_y^3 & 0 \\ -u_x^3 - u^3 u_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

In this case, we do not need any linear change of the independent variables  $x, y$ .

**Theorem 6.**  $\text{Rank}(g_\lambda) = 1$ . Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables and linear change of  $x$  and  $y$ , to one of the following forms:

$$P = \begin{pmatrix} d_x + \epsilon \left( u^2 d_y + \frac{u_y^2}{2} \right) & 0 & h u_y^2 \\ 0 & 0 & 0 \\ -h u_y^2 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x + f d_y + \frac{\partial_2 f u_y^2 + \partial_3 f u_y^3}{2} & 0 & -\frac{u_x^3 - h u_y^2 + f u_y^3}{u^1} \\ 0 & 0 & 0 \\ \frac{u_x^3 - h u_y^2 + f u_y^3}{u^1} & 0 & 0 \end{pmatrix}, \quad (14)$$

$$P = \begin{pmatrix} d_x + f d_y + \frac{\partial_2 f u_y^2 + \partial_3 f u_y^3}{2} & u_x^3 + h u_y^3 & 0 \\ -u_x^3 - h u_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_x + u^2 d_y + \frac{u_y^2}{2} & -\frac{u_x^2 + u^2 u_y^2}{u^1} & -\frac{u_x^3 + u^2 u_y^3}{u^1} \\ \frac{u_x^2 + u^2 u_y^2}{u^1} & 0 & 0 \\ \frac{u_x^3 + u^2 u_y^3}{u^1} & 0 & 0 \end{pmatrix}, \quad (15)$$

where  $f = f(u^2, u^3)$ ,  $h = h(u^2, u^3)$  are arbitrary functions and  $\epsilon$  can be either 0 or 1.

**Theorem 7.**  $\text{Rank}(g_\lambda) = 2$ . Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables and linear change of  $x$  and  $y$ , to one of the following forms:

$$P = \begin{pmatrix} -2u^1 d_y - u_y^1 & d_x + u^2 d_y + 2u_y^2 & \epsilon u_y^3 \\ d_x + u^2 d_y - u_y^2 & 0 & 0 \\ -\epsilon u_y^3 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x & d_y \\ d_x & 0 & 0 \\ d_y & 0 & 0 \end{pmatrix}, \quad (16)$$

$$P = \begin{pmatrix} p d_y + \frac{p' u_y^3}{2} & d_x + q d_y + \epsilon u_y^3 & 0 \\ d_x + q d_y + (q' - \epsilon) u_y^3 & r d_y + \frac{r' u_y^3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} d_y & d_x & -\frac{u_x^3}{u^2} \\ d_x & 0 & 0 \\ \frac{u_x^3}{u^2} & 0 & 0 \end{pmatrix}, \quad (17)$$

$$P = \begin{pmatrix} \epsilon d_y & d_x + u^3 d_y & -\frac{u_x^3 + u^3 u_y^3}{u^2} \\ d_x + u^3 d_y + u_y^3 & 0 & 0 \\ \frac{u_x^3 + u^3 u_y^3}{u^2} & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x & -\frac{u_x^3 - u^1}{u^2} \\ d_x & 0 & d_y \\ \frac{u_x^3 - u^1}{u^2} & d_y & 0 \end{pmatrix}, \quad (18)$$

$$P = \begin{pmatrix} 0 & d_x & d_y - \frac{u_x^3 - u_y^2}{u^2} \\ d_x & 0 & 0 \\ d_y + \frac{u_x^3 - u_y^2}{u^2} & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} u^1 d_y + \frac{u_y^1}{2} & d_x - \frac{u^2}{2} d_y - u_y^2 & -\frac{u_x^3}{u^2} \\ d_x - \frac{u^2}{2} d_y + \frac{u_y^2}{2} & 0 & 0 \\ \frac{u_x^3}{u^2} & 0 & 0 \end{pmatrix}, \quad (19)$$

$$P = \begin{pmatrix} d_y & d_x - u^3 d_y & \frac{u_x^3 - 2u^3 u_y^3}{u^3 u^1 - u^2} \\ d_x - u^3 d_y - u_y^3 & (u^3)^2 d_y + u^3 u_y^3 & -\frac{u^3 u_x^3 - 2(u^3)^2 u_y^3}{u^3 u^1 - u^2} \\ -\frac{u_x^3 - 2u^3 u_y^3}{u^3 u^1 - u^2} & \frac{u^3 u_x^3 - 2(u^3)^2 u_y^3}{u^3 u^1 - u^2} & 0 \end{pmatrix}, \quad (20)$$

$$P = \begin{pmatrix} \frac{\kappa d_y}{u^3} - \frac{\kappa u_y^3}{(u^3)^2} & d_x - \kappa d_y + \frac{\kappa u_y^3}{2u^3} & \frac{u_x^3 - 2\kappa u_y^3}{u^3 u^1 - u^2} \\ d_x - \kappa d_y - \frac{\kappa u_y^3}{2u^3} & \frac{\kappa u^3 d_y + u_y^3}{u^3 u^1 - u^2} & -\frac{u^3 u_x^3 - 2\kappa u^3 u_y^3}{u^3 u^1 - u^2} \\ -\frac{u_x^3 - 2\kappa u_y^3}{u^3 u^1 - u^2} & \frac{u^3 u_x^3 - 2\kappa u^3 u_y^3}{u^3 u^1 - u^2} & 0 \end{pmatrix}, \quad (21)$$

where  $p, q, r$  are arbitrary functions on  $u^3$ ,  $\kappa$  is constant and  $\epsilon$  can be either 0 or 1.

The proof of these theorems can be found in the Appendix.

Let us point out that, after swapping the coordinates  $u^1, u^2$ ,  $(18)_2$  corresponds to the Hamiltonian operator for the 2D equations of gas dynamic (see, for instance, [7]), namely

$$P^{ij} = \begin{pmatrix} 0 & d_x & d_y \\ d_x & 0 & \frac{u_y^2 - u_x^3}{u^1} \\ d_y & \frac{u_x^3 - u_y^2}{u^1} & 0 \end{pmatrix}, \quad (22)$$

We will discuss it in Section 3.2.

### 3 Hamiltonian systems of hydrodynamic type in 2+1 dimensions

In this section we discuss (2+1)-dimensional Hamiltonian systems of hydrodynamic type,

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0, \quad (23)$$

which are representable in the form  $\mathbf{u}_t + Ph_{\mathbf{u}} = 0$ , where  $h(\mathbf{u})$  is a Hamiltonian density and  $P$  is a two-dimensional Hamiltonian operator of differential-geometric type (6). As we recalled in the introduction, a (2+1)-dimensional quasilinear system is said to be integrable if it can be decoupled in infinitely many ways into a pair of compatible  $m$ -component one-dimensional systems in Riemann invariants. Let us briefly describe the method of hydrodynamic reduction introduced by Ferapontov and Khusnutdinova in [6].

#### 3.1 The method of hydrodynamic reductions

The method of hydrodynamic reductions is based on the existence of exact solutions of the (2+1)-dimensional system (23). These solutions have the form  $\mathbf{u} = \mathbf{u}(R^1, \dots, R^m)$ , where the Riemann invariants  $\mathbf{R} = (R^1, \dots, R^m)$  solve a pair of commuting diagonal systems

$$R_t^i = \lambda^i(\mathbf{R}) R_x^i, \quad R_y^i = \mu^i(\mathbf{R}) R_x^i. \quad (24)$$

Let us point out that we do not impose any constraint on the number of Riemann invariants:  $m$  is arbitrary. Therefore, the (2+1)-dimensional system we are considering (23), is decoupled into a pair of diagonal (1+1)-dimensional systems given by (24). Usually, these solutions are known as nonlinear interactions of  $m$  planar simple waves.

It turns out that the commutativity of the flows (24) is equivalent to the following constraints on the characteristic speeds  $\lambda^i, \mu^i$  [20]:

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_j = \frac{\partial}{\partial R^j}, \quad (25)$$

(no summation). Imposing these restrictions, the general solution of systems (24) is given by the implicit generalised hodograph formula [20]

$$v^i(\mathbf{R}) = x + \lambda^i(\mathbf{R}) t + \mu^i(\mathbf{R}) y, \quad i = 1, \dots, m. \quad (26)$$

Here the functions  $v^i(\mathbf{R})$  are characteristic speeds of the general flow commuting with (24), namely, the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j. \quad (27)$$

By straightforward computation, the substitution of  $\mathbf{u}(R^1, \dots, R^m)$  into (23), using (24), leads to

$$(E + \lambda^i A + \mu^i B) \partial_i \mathbf{u} = 0, \quad i = 1, \dots, m, \quad (28)$$

where  $E$  is the  $n \times n$  identity matrix. This means that both  $\lambda^i$  and  $\mu^i$  have to satisfy the dispersion relation

$$\det(E + \lambda^i A + \mu^i B) = 0. \quad (29)$$

Furthermore, the construction of nonlinear interactions of  $m$  planar simple waves can be summarised as follows. First of all, we have to decouple the initial (2+1)-dimensional system (23) into a pair of commuting flows (24), by solving the equations (25), (28) for  $\mathbf{u}(\mathbf{R})$ ,  $\lambda^i(\mathbf{R})$ ,  $\mu^i(\mathbf{R})$  as functions depending on the Riemann invariants  $R^1, \dots, R^m$ . It is not difficult to see that for  $m \geq 3$  the system given by these equations is overdetermined. Thus, in general this system does not possess solutions. However, if we are able to construct a particular reduction of the form (24), the second step is quite straightforward: we have to solve the linear system given by (27) for the functions  $v^i(\mathbf{R})$ , and then we can obtain  $R^1, \dots, R^m$  as functions of  $t, x, y$  from the implicit hodograph formula (26).

What can we say about the number of  $m$ -component reductions that a (2+1)-dimensional system (23) may admit? Analysing equations (25) and (28), one can prove that this number is parametrised, up to changes of variables of the form  $R^i \rightarrow f^i(R^i)$ , by  $m$  arbitrary functions of a single variable. Remarkably, this number does not depend on  $n$ . This leads to the following definition.

**Definition 2** ([6]). *A (2+1)-dimensional quasilinear system is said to be integrable if it possesses  $m$ -component reductions of the form (24) parametrised by  $m$  arbitrary functions of a single argument.*

**Remark.** Looking at the structure of equations (25) and (28), one can see that their consistency conditions involve only triple of indices  $i \neq j \neq k$ . Moreover, all these conditions are completely symmetric in  $i, j$  and  $k$ , and then it is enough to verify them setting, for instance,  $i = 1, j = 2, k = 3$ . This means that the existence of non-trivial three-component reductions implies the existence of  $m$ -component reductions for arbitrary  $m$ .

**Remark.** We require that  $\lambda^i$  and  $\mu^i$  do not satisfy any linear relation, otherwise we would have no sufficiently many arbitrary functions of a single argument. Indeed, let us suppose that  $\mu^i = a\lambda^i + b$ . Condition (25) reads  $\partial_j a \lambda^i + \partial_j b = 0$ , which implies  $a$  and  $b$  constant. Thus, solutions of the system (24),

$$R_t^i = \lambda^i R_x^i, \quad R_y^i = (a\lambda^i + b) R_x^i,$$

are of the form  $R^i = R^i(x + by, t + ay)$ . These solutions correspond to travelling wave reduction, and they clearly do not contain enough arbitrary functions.

### 3.2 Generalised two-dimensional gas dynamic equations

The equations of two-dimensional isentropic gas dynamics are of the form

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \quad u_t + uu_x + uv_y + \frac{p_x}{\rho} = 0, \quad v_t + uv_x + vv_y + \frac{p_y}{\rho} = 0, \quad (30)$$

where  $p = p(\rho)$  is the equation of state. In matrix form (23), one has  $\mathbf{u} = (\rho, u, v)^t$  and

$$A = \begin{pmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad B = \begin{pmatrix} v & 0 & \rho \\ 0 & v & 0 \\ \frac{c^2}{\rho} & 0 & v \end{pmatrix},$$



where  $c^2 = p'(\rho)$  is the sound speed. As demonstrated in [17], there exist potential flows describing nonlinear interaction of two sound waves which are locally parametrised by four arbitrary functions of a single argument.

The system (30) can be written in Hamiltonian form as  $\mathbf{u}_t + Ph_{\mathbf{u}} = 0$ , where the operator  $P$  is given by (22), namely

$$P^{ij} = \begin{pmatrix} 0 & d_x & d_y \\ d_x & 0 & \frac{u_y - v_x}{\rho} \\ d_y & \frac{v_x - u_y}{\rho} & 0 \end{pmatrix},$$

the Hamiltonian density  $h$  is  $h(\rho, u, v) = \frac{1}{2}\rho(u^2 + v^2) + k(\rho)$ , and the equation of state  $p$  and the function  $k$  are related by  $p_\rho = \rho k_{\rho\rho}$ .

Let us assume  $h = h(\rho, u, v)$  generic, thus the system  $\mathbf{u}_t + Ph_{\mathbf{u}} = 0$  reads

$$\rho_t + (h_u)_x + (h_v)_y = 0, \quad u_t + (h_\rho)_x + \frac{u_y - v_x}{\rho} h_v = 0, \quad v_t + (h_\rho)_y + \frac{v_x - u_y}{\rho} h_u = 0. \quad (31)$$

Let us consider the Riemann invariants  $R^1, \dots, R^m$  solving

$$R_x^i = \lambda^i(\mathbf{R}) R_t^i, \quad R_y^i = \mu^i(\mathbf{R}) R_t^i, \quad i = 1, \dots, m.$$

By straightforward computation, the substitution  $\rho = \rho(\mathbf{R}), u = u(\mathbf{R}), v = v(\mathbf{R})$  into (31) implies

$$(1 + \lambda^i h_{\rho u} + \mu^i h_{\rho v}) \partial_i u + h_{\rho\rho} \lambda^i \partial_i \rho = 0, \quad (32)$$

$$(1 + \lambda^i h_{\rho u} + \mu^i h_{\rho v}) \partial_i v + h_{\rho\rho} \mu^i \partial_i \rho = 0, \quad (33)$$

$$(1 + \lambda^i h_{\rho u} + \mu^i h_{\rho v}) \partial_i \rho + (\lambda^i h_{uu} + \mu^i h_{uv}) \partial_i u + (\lambda^i h_{uv} + \mu^i h_{vv}) \partial_i v = 0, \quad (34)$$

here  $i = 1, \dots, m$ ,  $\partial_i = \frac{\partial}{\partial R^i}$ . Note that since  $\mu^i \partial_i u = \lambda^i \partial_i v$  (this easy follows from (32) and (33), assuming  $1 + \lambda^i h_{\rho u} + \mu^i h_{\rho v} \neq 0$ ), one has  $u_y = v_x$ . Thus, solutions are necessarily potential. Then, setting  $u = \varphi_x$  and  $v = \varphi_y$ , our system (31) reads

$$\rho_t + (h_u)_x + (h_v)_y = 0, \quad \varphi_{xt} + (h_\rho)_x = 0, \quad \varphi_{yt} + (h_\rho)_y = 0. \quad (35)$$

Both the last two equations give  $\varphi_t + h_\rho = 0$ , so we finally have the following system

$$\rho_t + (h_u)_x + (h_v)_y = 0, \quad \varphi_t + h_\rho = 0. \quad (36)$$

If we consider the partial Legendre transform

$$\tilde{\rho} = h_\rho, \quad \tilde{u} = u, \quad \tilde{v} = v, \quad \tilde{h} = h - \rho h_\rho, \quad (37)$$

the derivatives respect the new variables are

$$\tilde{h}_{\tilde{\rho}} = -\rho, \quad \tilde{h}_{\tilde{u}} = h_u, \quad \tilde{h}_{\tilde{v}} = h_v, \quad (38)$$

and we can rewrite the system (36) in the form

$$(\tilde{h}_{\tilde{\rho}})_t + (\tilde{h}_{\tilde{u}})_x + (\tilde{h}_{\tilde{v}})_y = 0, \quad \varphi_t = \tilde{\rho}, \quad \varphi_x = \tilde{u}, \quad \varphi_y = \tilde{v}.$$

The function  $\tilde{h}$  depends only on  $\varphi_x, \varphi_y, \varphi_t$  and thus we obtain three-dimensional Euler-Lagrange equations (setting  $\tilde{h} = f$ )

$$(f_{\varphi_x})_x + (f_{\varphi_y})_y + (f_{\varphi_t})_t = 0, \quad (39)$$

corresponding to Lagrangian densities of the form  $f(\varphi_x, \varphi_y, \varphi_t)$ . For example, the Lagrangian density  $f = u_x^2 + u_y^2 - 2e^{ut}$  leads to the Boyer-Finley equation  $u_{xx} + u_{yy} = e^{ut}u_{tt}$  [3].

In [8] Ferapontov, Khusnutdinova and Tsarev derived a system of partial differential equations for the Lagrangian density  $f(\varphi_x, \varphi_y, \varphi_t)$  which are necessary and sufficient for the integrability of the equation (39) by the method of hydrodynamic reductions (see also [10] for further details). Setting  $a = \varphi_x, b = \varphi_y, c = \varphi_t$ , these conditions can be represented in a remarkable compact form:

**Theorem 8** ([8]). *For a non-degenerate Lagrangian, the Euler-Lagrange equation (39) is integrable by the method of hydrodynamic reductions if and only if the density  $f$  satisfies the relation*

$$d^4 f = d^3 f \frac{dH}{H} + \frac{3}{H} \det(dM); \quad (40)$$

here  $d^3 f$  and  $d^4 f$  are the symmetric differentials of  $f$ . The Hessian  $H$  and the  $4 \times 4$  matrix  $M$  are defined as follows:

$$H = \det \begin{pmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ab} & f_{bb} & f_{bc} \\ f_{ac} & f_{bc} & f_{cc} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_a & f_b & f_c \\ f_a & f_{aa} & f_{ab} & f_{ac} \\ f_b & f_{ab} & f_{bb} & f_{bc} \\ f_c & f_{ac} & f_{bc} & f_{cc} \end{pmatrix}. \quad (41)$$

The differential  $dM = M_a da + M_b db + M_c dc$  is a matrix-valued form

$$\begin{aligned} dM = & \begin{pmatrix} 0 & f_{aa} & f_{ab} & f_{ac} \\ f_{aa} & f_{aaa} & f_{aab} & f_{aac} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \end{pmatrix} da + \begin{pmatrix} 0 & f_{ab} & f_{bb} & f_{bc} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{bb} & f_{abb} & f_{bbb} & f_{bbc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \end{pmatrix} db \\ & + \begin{pmatrix} 0 & f_{ac} & f_{bc} & f_{cc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \\ f_{cc} & f_{acc} & f_{bcc} & f_{ccc} \end{pmatrix} dc. \end{aligned}$$

Finally, we recall that the equations of gas dynamic possess only double waves reduction, and are not integrable by the method of hydrodynamic reductions [7]. On the other hand, the generalised equations (31) define a (2+1)-dimensional integrable system when the Lagrangian density  $f(\varphi_x, \varphi_y, \varphi_t)$ , obtained by the Hamiltonian density  $h(\rho, u, v)$  performing a partial Legendre transform (37), satisfies the conditions given by Theorem 8.

### 3.3 Three-component Hamiltonian systems with degenerate structure

We have seen that the degenerate Hamiltonian operator  $(18)_2$  leads to a class of integrable systems related to the Lagrangian density of the form  $f(\varphi_x, \varphi_y, \varphi_t)$ . Here we are going to describe all three-component cases arising from our classification.

The aim of this section is to apply the method of hydrodynamic reductions to three-component Hamiltonian systems given by  $\mathbf{u}_t + P\mathbf{h}_{\mathbf{u}} = 0$ , where  $P$  is a Hamiltonian structure appearing in Theorems 5, 6 and 7. Let us identify the Hamiltonian operators we obtained with the rank of the pencil  $g_\lambda$ . For instance, we call rank-zero structures the Hamiltonian operators listed in Theorem 5.

**Theorem 9.** *The method of hydrodynamic reductions imposes additional differential constraints under which equations under study reduce to known classes of systems considered before:*

- rank-zero structures lead to trivial systems

$$u_t^1 = u_t^2 = u_t^3 = 0,$$

- rank-one structures lead to one dimensional system of the form

$$u_t^1 + f(u^1)u_x^1 = 0, \quad u_t^2 = u_t^3 = 0,$$

- rank-two structures lead either to one dimensional system to the form

$$u_t^1 + (h_{u^2})_x = 0, \quad u_t^2 + (h_{u^1})_x = 0, \quad u_t^3 = 0,$$

or two-component non-degenerate Hamiltonian systems

$$u_t^1 + (h_{u^1})_x = 0, \quad u_t^2 + (h_{u^2})_y = 0, \quad (42)$$

$$u_t^1 + (h_{u^2})_x = 0, \quad u_t^2 + (h_{u^1})_x + (h_{u^2})_y = 0, \quad (43)$$

$$u_t^1 + (2u^1 h_{u^1} + u^2 h_{u^2} - h)_x + (u^1 h_{u^2})_y = 0, \quad u_t^2 + (u^2 h_{u^1})_x + (2u^2 h_{u^2} + u^1 h_{u^1} - h)_y = 0, \quad (44)$$

plus the trivial equation  $u_t^3 = 0$ , or to the system

$$u_t^1 + (h_{u^2})_x + (h_{u^3})_y = 0, \quad u_t^2 + (h_{u^1})_x = 0, \quad u_t^3 + (h_{u^1})_y = 0. \quad (45)$$

We point out that the integrability of two-component non-degenerate Hamiltonian systems (42), (43) and (44), generated respectively by the Hamiltonian operators

$$P = \begin{pmatrix} d_x & 0 \\ 0 & d_y \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x \\ d_x & d_y \end{pmatrix}, \quad P = \begin{pmatrix} 2u^1 & u^2 \\ u^2 & 0 \end{pmatrix} d_x + \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix} d_y + \begin{pmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{pmatrix},$$

is completely understood, see [11] for further details. Furthermore, as we showed above, system (45) reduces to the three-dimensional Euler-Lagrange equations (39) after performing a partial Legendre transformation of the form (37).

### Proof of Theorem 9:

First of all, let us remark that if  $u_t^i = 0$ , for some  $i$ , the method of hydrodynamic reductions necessarily implies  $u^i = \text{const}$ . Secondly, if one of the equations of the system is of the form  $u_t^i + \phi(\mathbf{u})u_x^i + \psi(\mathbf{u})u_y^i = 0$ , the method of hydrodynamic reductions implies  $(\lambda^j + \phi + \psi\mu^j)\partial_j u^i = 0$ , which leads to  $u^i = \text{const}$ , since we are imposing that  $\lambda^j$  and  $\mu^j$  do not satisfy any linear relation. Furthermore, in these cases we can replace  $u^i$  with a constant, and then the Hamiltonian will depend on  $u^j$  for  $j \neq i$ .

Using these observations, the proof is straightforward. Rank-zero structures easily lead to trivial systems. For the rank-one structures we always have  $u^2$  and  $u^3$  constant, which leads to an operator of the form

$$P = \begin{pmatrix} d_x + \kappa d_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa = \text{const},$$

which is essentially one-dimensional (up to linear change of the independent variables  $x$  and  $y$ ).

The analysis of rank-two structures is a bit more complicated. In the cases  $(16)_1$  and  $(19)_2$ , the method of hydrodynamic reductions implies  $u^3 = \text{const}$ . Thus, up to a change of local coordinates  $u^1, u^2$ , the  $3 \times 3$  degenerate Hamiltonian operator reduces to direct sum of the  $2 \times 2$  two-component non-degenerate Mokhov's Hamiltonian operator [13, 14]

$$P = \begin{pmatrix} 2u^1 & u^2 \\ u^2 & 0 \end{pmatrix} d_x + \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix} d_y + \begin{pmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{pmatrix},$$

and the trivial  $1 \times 1$  operator  $P = 0$ .

In the cases  $(17)_{1,2}$ ,  $(18)_1$ ,  $(20)$  and  $(21)$  the method of hydrodynamic reductions implies again  $u^3 = \text{const}$ . These structures reduce to direct sum of constant  $2 \times 2$  two-component non-degenerate Hamiltonian operator, and the trivial  $1 \times 1$  operator  $P = 0$ . Constant  $2 \times 2$  non-degenerate Hamiltonian operators are known [11]: if they do not reduce to one-dimensional operator

$$P = \begin{pmatrix} 0 & d_x \\ d_x & 0 \end{pmatrix},$$

(for instance,  $(18)_1$  for  $\epsilon = 1$ ), they can be brought to one of the following two forms

$$P = \begin{pmatrix} d_x & 0 \\ 0 & d_y \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d_x \\ d_x & d_y \end{pmatrix},$$

by a change of local coordinates  $u^1, u^2$  and a linear change of the independent variables  $x, y$ .

It remains to consider the cases  $(16)_2$  and  $(19)_1$ . It is not difficult to see that in both cases we get  $u_y^2 = u_x^3$ . Therefore, solutions are necessarily potential. Then, setting  $u^2 = \varphi_x$  and  $u^3 = \varphi_y$ , the system leads to (35). ■

## 4 Concluding remarks

The problem of classification of 2D Hamiltonian operators of differential geometric-type, proposed by Dubrovin and Novikov in [5], is now completely solved up to three-component case. Even though in [9] we provided a complete classification of non-degenerate operators up to four components, in the degenerate case it is still open. The main obstacle is the lack of a full description of one-dimensional degenerate Poisson brackets. Indeed, already for four-component one-dimensional degenerate structures, the computation of Jacobi conditions is quite complicated [12, 16].

As we have said above, any 2D degenerate Hamiltonian operator gives rise to a pair of 1D compatible degenerate brackets of Dubrovin-Novikov type (see [14, 15] for further details). Some of the degenerate bi-Hamiltonian structures arising from our classification are not of the kind investigated by Strachan [18, 19]. It would be interesting to analyse these structures and to study a possible correspondence with the analogous of Frobenius manifolds with degenerate metric.

## Acknowledgments

I would like to thank Eugene Ferapontov and Paolo Lorenzoni for useful remarks.

## Appendix. Proof of Theorems 5, 6 and 7

### Proof of Theorem 5

In the case where the pencil  $g_\lambda$  has rank constantly equal to 0, both the metrics must be identically null. Thus, by Theorem 3, we can always reduce the coefficients  $b_k^{ij}$  to  $b_3^{12} = -b_3^{21} = 1$ , that is, the  $x$ -part of the 2D Hamiltonian operator can be fixed as (9). Imposing (5), we get that all the coefficients  $\tilde{b}_k^{ij}$  must vanish except  $\tilde{b}_3^{12} = -\tilde{b}_3^{21} = \nu(u^1, u^2, u^3)$ . The transformations which preserve the form of  $P_{(x)}$  have the form

$$u^1 = \varphi^1(v^1, v^2, v^3), \quad u^2 = \varphi^2(v^1, v^2, v^3), \quad u^3 = \varphi^3(v^3),$$

with the constraint

$$\partial_1 \varphi^1 \partial_2 \varphi^2 - \partial_2 \varphi^1 \partial_1 \varphi^2 = (\varphi^3)'$$

If  $\nu = \xi$  is constant, we get  $\tilde{b} = \xi b$ . Let us assume that  $\nu = \nu(u^3)$ . Then, in the new system of coordinates, it is always possible to reduce  $\nu$  to  $v^3$ . Let us finally assume that  $\nu$  is an arbitrary function of  $u^1, u^2, u^3$ . Then, there exists a change of coordinate preserving  $P_{(x)}$  which transforms  $\nu$  to  $v^1$  or, equivalently, to  $v^2$  (these two cases are the same since we can swap  $v^1, v^2$ ). Summarising,  $P_{(y)}$  leads to one of the following two structures

$$P_{(y)} = \begin{pmatrix} 0 & v^1 v_y^3 & 0 \\ -v^1 v_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_{(y)} = \begin{pmatrix} 0 & v^3 v_y^3 & 0 \\ -v^3 v_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are not equivalent modulo transformations which preserve the form of  $P_{(x)}$ .

### Proof of Theorem 6

When the rank of the pencil  $g^{ij} + \lambda \tilde{g}^{ij}$  is constantly equal to one, we can always work in a coordinate system where the metrics assume the forms

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $f = f(u^1, u^2, u^3)$ . Let us now consider separately each case given by Theorem 3.

**Case (10)<sub>1</sub>.** The symbols  $b_k^{ij}$  are identically 0. In this case, a generic transformation which preserves  $P_{(x)}$  is given by

$$u^1 = v^1 + \varphi^1(v^2, v^3), \quad u^2 = \varphi^2(v^2, v^3), \quad u^3 = \varphi^3(v^2, v^3). \quad (46)$$

Let us point out that this change of coordinates transforms  $b_k^{ij}$  (and then  $\tilde{b}_k^{ij}$ ) as components of a  $(2, 1)$ -tensor [16]. Conditions (5) imply two solutions.

*Solution 1.* The first solution reads

$$f = f(u^2, u^3), \quad \tilde{b}_2^{11} = \frac{\partial_2 f}{2}, \quad \tilde{b}_3^{11} = \frac{\partial_3 f}{2}, \quad \tilde{b}_2^{21} = -\tilde{b}_2^{12} = \tilde{b}_3^{13} = -\tilde{b}_3^{31} = \psi, \\ \tilde{b}_3^{21} = -\tilde{b}_3^{12} = \frac{\psi^2}{\eta}, \quad \tilde{b}_2^{13} = -\tilde{b}_2^{31} = \eta,$$

where  $\psi = \psi(u^2, u^3)$  and  $\eta = \eta(u^2, u^3)$ , and all other  $b_k^{ij}$  vanish. Clearly, we are imposing  $\eta \neq 0$ . The operator leads to

$$P_{(y)} = \begin{pmatrix} f d_y + \frac{\partial_2 f u_y^2 + \partial_3 f u_y^3}{2} & -\psi u_y^2 - \frac{\psi^2}{\eta} u_y^3 & \eta u_y^2 + \psi u_y^3 \\ \psi u_y^2 + \frac{\psi^2}{\eta} u_y^3 & 0 & 0 \\ -\eta u_y^2 - \psi u_y^3 & 0 & 0 \end{pmatrix},$$

where  $f = f(u^2, u^3)$ ,  $\psi = \psi(u^2, u^3)$  and  $\eta = \eta(u^2, u^3) \neq 0$ . Applying a transformation of the form (46) we get  $f \rightarrow f(\varphi^2, \varphi^3)$  and

$$\tilde{b}_2^{21} = -\tilde{b}_2^{12} = \tilde{b}_3^{13} = -\tilde{b}_3^{31} = \psi \rightarrow \frac{(\partial_3 \varphi^2 \eta + \partial_3 \varphi^3 \psi)(\partial_2 \varphi^2 \eta + \partial_2 \varphi^3 \psi)}{(\partial_2 \varphi^2 \partial_3 \varphi^3 - \partial_3 \varphi^2 \partial_2 \varphi^3) \eta}, \\ \tilde{b}_3^{21} = -\tilde{b}_3^{12} = \frac{\psi^2}{\eta} \rightarrow -\frac{(\partial_3 \varphi^2 \eta + \partial_3 \varphi^3 \psi)^2}{(\partial_2 \varphi^2 \partial_3 \varphi^3 - \partial_3 \varphi^2 \partial_2 \varphi^3) \eta}, \\ \tilde{b}_2^{13} = -\tilde{b}_2^{31} = \eta \rightarrow \frac{(\partial_2 \varphi^2 \eta + \partial_2 \varphi^3 \psi)^2}{(\partial_2 \varphi^2 \partial_3 \varphi^3 - \partial_3 \varphi^2 \partial_2 \varphi^3) \eta}.$$

We cannot choose both  $\partial_2\varphi^2\eta + \partial_2\varphi^3\psi = 0$  and  $\partial_3\varphi^2\eta + \partial_3\varphi^3\psi = 0$ , otherwise we would have the denominator equal to zero. However, a suitable choice of the functions  $\varphi^2$  and  $\varphi^3$  allows us to reduce  $f$  to either  $v^2$  or  $v^3$  (which are equivalent up to swapping  $v^2$  and  $v^3$ ) if  $f$  is not constant, and  $\psi$  to zero. This leads to two operators

$$P_{(y)} = \begin{pmatrix} \kappa d_y & 0 & \tilde{\eta}v_y^2 \\ 0 & 0 & 0 \\ -\tilde{\eta}v_y^2 & 0 & 0 \end{pmatrix}, \quad P_{(y)} = \begin{pmatrix} v^2 d_y + \frac{v_y^2}{2} & 0 & \tilde{\eta}v_y^2 \\ 0 & 0 & 0 \\ -\tilde{\eta}v_y^2 & 0 & 0 \end{pmatrix},$$

where  $\kappa$  is constant and  $\tilde{\eta} = \tilde{\eta}(v^2, v^3)$ . Allowing linear change of  $x$  and  $y$ ,  $\kappa$  can be brought to zero.

*Solution 2.* In the case where  $\eta = 0$ , the solution reads

$$f = f(u^2, u^3), \quad \tilde{b}_2^{11} = \frac{\partial_2 f}{2}, \quad \tilde{b}_3^{11} = \frac{\partial_3 f}{2}, \quad \tilde{b}_3^{12} = -\tilde{b}_3^{21} = \nu(u^2, u^3),$$

and all other  $b_k^{ij}$  vanish. Modulo transformations of the form (46) this case corresponds to the previous one.

**Case (10)<sub>2</sub>.** Here  $b_3^{12} = -b_3^{21} = 1$ , while other symbols  $b_k^{ij}$  are identically 0. Conditions (5) imply

$$f = f(u^2, u^3), \quad \tilde{b}_2^{11} = \frac{\partial_2 f}{2}, \quad \tilde{b}_3^{11} = \frac{\partial_3 f}{2}, \quad \tilde{b}_3^{12} = -\tilde{b}_3^{21} = \nu(u^2, u^3),$$

and other  $\tilde{b}_k^{ij} = 0$ . A generic transformation which preserves  $P_{(x)}$  is given by

$$u^1 = v^1, \quad u^2 = \partial_3\varphi^3(v^3)v^2 + \varphi^2(v^3), \quad u^3 = \varphi^3(v^3).$$

Unfortunately, in general this group of transformations cannot help to simplify our structure (the operator depends on two functions of  $u^2, u^3$ , while the group depends only on two functions of  $u^3$ ). We should consider separately each case where the functions  $f$  and  $\nu$  are constant or depend on one single variables. Therefore, it is more reasonable to consider just the general solution, namely

$$P_{(y)} = \begin{pmatrix} f d_y + \frac{\partial_2 f u_y^2 + \partial_3 f u_y^3}{2} & \nu u_y^3 & 0 \\ -\nu u_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for arbitrary  $f = f(u^2, u^3)$ ,  $\nu = \nu(u^2, u^3)$ .

**Case (10)<sub>3</sub>.** Here  $b_3^{31} = -b_3^{13} = \frac{1}{u^1}$ , while other symbols  $b_k^{ij}$  are identically 0. Conditions (5) imply

$$f = f(u^2, u^3), \quad \tilde{b}_2^{11} = \frac{\partial_2 f}{2}, \quad \tilde{b}_3^{11} = \frac{\partial_3 f}{2}, \quad \tilde{b}_3^{31} = -\tilde{b}_3^{13} = \frac{f}{u^1}, \quad \tilde{b}_2^{13} = -\tilde{b}_2^{31} = \frac{\nu}{u^1},$$

where  $\nu = \nu(u^2, u^3)$ , and other  $\tilde{b}_k^{ij} = 0$ . A generic transformation which preserves  $P_{(x)}$  is given by

$$u^1 = v^1, \quad u^2 = \varphi^2(v^2), \quad u^3 = \varphi^3(v^3).$$

As before, this group of transformations cannot help to simplify our structure for arbitrary  $f$  and  $\nu$ . Therefore, the operator leads to

$$P_{(y)} = \begin{pmatrix} f d_y + \frac{\partial_2 f u_y^2 + \partial_3 f u_y^3}{2} & 0 & \frac{\nu u_y^2 - f u_y^3}{u^1} \\ 0 & 0 & 0 \\ -\frac{\nu u_y^2 - f u_y^3}{u^1} & 0 & 0 \end{pmatrix},$$

for arbitrary  $f = f(u^2, u^3)$ ,  $\nu = \nu(u^2, u^3)$ .

**Case (10)<sub>4</sub>.** Here  $b_2^{21} = -b_2^{12} = -b_3^{13} = b_3^{31} = \frac{1}{u^1}$ , while other symbols  $b_k^{ij}$  are identically 0. Conditions (5) imply

$$f = f(u^2, u^3), \quad \tilde{b}_2^{11} = \frac{\partial_2 f}{2}, \quad \tilde{b}_3^{11} = \frac{\partial_3 f}{2}, \quad \tilde{b}_2^{21} = -\tilde{b}_2^{12} = -\tilde{b}_3^{13} = \tilde{b}_3^{31} = \frac{f}{u^1},$$

and other  $\tilde{b}_k^{ij} = 0$ . A generic transformation which preserves  $P_{(x)}$  is given by

$$u^1 = v^1, \quad u^2 = \varphi^2(v^2, v^3), \quad u^3 = \varphi^3(v^2, v^3).$$

This change of coordinates transforms the objects  $\tilde{b}_k^{ij}$  as components of a  $(2, 1)$ -tensor [16]. If  $f = \xi$  is constant, then  $\tilde{g} = \xi g$  and  $\tilde{b} = \xi b$ . Otherwise, we can choose  $\varphi^2$  or  $\varphi^3$  such that  $f$  reduce either to  $v^2$  or  $v^3$ , which are equivalent forms since we can swap  $v^2, v^3$ . Thus  $P_{(y)}$  leads to

$$P_{(y)} = \begin{pmatrix} v^2 d_y + \frac{v_y^2}{2} & -\frac{v^2 v_y^2}{v^1} & -\frac{v^2 v_y^3}{v^1} \\ \frac{v^2 v_y^2}{v^1} & 0 & 0 \\ \frac{v^2 v_y^3}{v^1} & 0 & 0 \end{pmatrix}.$$

### Proof of Theorem 7

When the rank of the pencil  $g^{ij} + \lambda \tilde{g}^{ij}$  is constantly equal to two, we have three possibilities:

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} p & q & 0 \\ q & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (47)$$

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} p & q & r \\ q & 0 & 0 \\ r & 0 & 0 \end{pmatrix}, \quad (48)$$

or

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{g}^{ij} = \begin{pmatrix} 0 & q & 0 \\ q & p & r \\ 0 & r & 0 \end{pmatrix}, \quad (49)$$

where  $p, q, r$  are arbitrary functions of  $u^1, u^2, u^3$ . We remark that (48) and (49) are equivalent up to a transformation of the form  $u^1 = v^2, u^2 = v^1, u^3 = v^3$  (which preserves the form of the first metric). However, this change of coordinate does not fix the structures  $(11)_2$  and  $(11)_3$ . Therefore, when the  $x$ -part of the operator is given by  $(11)_1$ , we can avoid (49), while in the other two cases,  $(11)_2$  and  $(11)_3$ , we have to take it into account.

**Case (11)<sub>1</sub>.** Here the first structure is  $(11)_1$ , and the group of transformations which preserve its form is given by

$$u^1 = e^\psi v^1 + \varphi^1(v^3), \quad u^2 = e^{-\psi} v^2 + \varphi^2(v^3), \quad u^3 = \varphi^3(v^3), \quad (50)$$

where  $\psi$  is constant, plus the switch of  $u^1, u^2$  (note that this change of coordinates transforms the objects  $\tilde{b}_k^{ij}$  as components of a  $(2, 1)$ -tensor [16]). In the case where we are dealing with (47), up to swapping  $u^1, u^2$ , solutions of conditions (5) can be summarised as follows.

*Solution 1.* The first solution is given by

$$p = p(u^3) + \kappa u^1, \quad q = q(u^3) - \frac{\kappa u^2}{2}, \quad \tilde{b}_1^{11} = \tilde{b}_2^{21} = \frac{\kappa}{2}, \quad \tilde{b}_3^{11} = \frac{p'}{2},$$

$$\tilde{b}_2^{12} = -\kappa, \quad \tilde{b}_3^{12} = 2q', \quad \tilde{b}_3^{21} = -q',$$

where  $\kappa \neq 0$  is constant. This leads to

$$P_{(y)} = \begin{pmatrix} (p + \kappa u^1) d_y + \frac{\kappa u_y^1 + p' u_y^3}{2} & \left(q - \frac{\kappa u^2}{2}\right) d_y - \kappa u_y^2 + 2q' u_y^3 & 0 \\ \left(q - \frac{\kappa u^2}{2}\right) d_y + \frac{\kappa u_y^2}{2} - q' u_y^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A change of coordinates of the form (50) transforms

$$p \rightarrow \varphi^1 \kappa + p(\varphi^3), \quad q \rightarrow -\frac{\varphi^2 \kappa}{2} + q(\varphi^3), \quad \kappa \rightarrow \kappa e^{-\psi}$$

thus, it is always possible to choose  $\varphi^1, \varphi^2$  and  $\psi$  such that in the new coordinates  $p = q = 0$  and  $\kappa$  is fixed, let us set it equal to  $-2$ . Therefore, the operator leads to

$$P_{(y)} = \begin{pmatrix} -2v^1 d_y - v_y^1 & v^2 d_y + 2v_y^2 & 0 \\ v^2 d_y - v_y^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us point out that in this case the 2D operator  $P$  can be view as direct sum of  $2 \times 2$  Mokhov's operator [14]

$$P = \begin{pmatrix} -2v^1 d_y - v_y^1 & d_x + v^2 d_y + 2v_y^2 \\ d_x + v^2 d_y - v_y^2 & 0 \end{pmatrix}.$$

and trivial  $1 \times 1$  operator  $P = 0$ .

*Solution 2.* The second solution is given by

$$p = p(u^3) + \kappa u^1, \quad q = q(u^3) - \frac{\kappa u^2}{2}, \quad \tilde{b}_1^{11} = \tilde{b}_2^{21} = \tilde{b}_3^{31} = -\tilde{b}_3^{13} = \frac{\kappa}{2}, \quad \tilde{b}_3^{11} = \frac{p'}{2},$$

$$\tilde{b}_2^{12} = -\kappa, \quad \tilde{b}_3^{12} = q',$$

where  $\kappa \neq 0$  is constant. This leads to

$$P_{(y)} = \begin{pmatrix} (p + \kappa u^1) d_y + \frac{\kappa u_y^1 + p' u_y^3}{2} & \left(q - \frac{\kappa u^2}{2}\right) d_y - \kappa u_y^2 + q' u_y^3 & -\frac{\kappa u_y^3}{2} \\ \left(q - \frac{\kappa u^2}{2}\right) d_y + \frac{\kappa u_y^2}{2} & 0 & 0 \\ \frac{\kappa u_y^3}{2} & 0 & 0 \end{pmatrix}.$$

The group (50) acts on this case as the previous one. Thus, we can reduce  $p$  and  $q$  to zero, and  $\kappa$  to  $-2$ , obtaining

$$P_{(y)} = \begin{pmatrix} -2v^1 d_y - v_y^1 & v^2 d_y + 2v_y^2 & v_y^3 \\ v^2 d_y - v_y^2 & 0 & 0 \\ -v_y^3 & 0 & 0 \end{pmatrix}.$$

*Solution 3.* In the case where  $\kappa = 0$ , the solution is given by

$$p = p(u^3), \quad q = q(u^3), \quad r = r(u^3), \quad \tilde{b}_3^{11} = \frac{p'}{2}, \quad \tilde{b}_3^{12} = \nu(u^3), \quad \tilde{b}_3^{21} = q' - \nu, \quad \tilde{b}_3^{22} = \frac{r'}{2}.$$



This leads to

$$P_{(y)} = \begin{pmatrix} p d_y + \frac{p' u_y^3}{2} & q d_y + \nu u_y^3 & 0 \\ q d_y + (q' - \nu) u_y^3 & r d_y + \frac{r' u_y^3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, the group (50) acts on the objects as

$$p \rightarrow p(\varphi^3) e^{-2\psi}, \quad q \rightarrow q(\varphi^3), \quad r \rightarrow r(\varphi^3) e^{2\psi}, \quad \nu \rightarrow (\varphi^3)' \nu(\varphi^3).$$

Here we have four arbitrary functions  $p, q, r, \nu$  and only one function  $\varphi^3$  and one constant  $\psi$  acting on them. Thus, even if we could consider several cases (where some functions are constant or zero), it does not simplify the classification. However, let us make a choice: if  $\nu$  is non-zero, it can be always reduced to 1. Thus the operator leads to

$$P_{(y)} = \begin{pmatrix} p d_y + \frac{p' u_y^3}{2} & q d_y + \epsilon u_y^3 & 0 \\ q d_y + (q' - \epsilon) u_y^3 & r d_y + \frac{r' u_y^3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (51)$$

with  $\epsilon$  equal either to 0 or 1.

Let us deal with (48). If  $r = 0$ , solutions of conditions (5) lead to (51) (replacing  $r$  with 0). Otherwise, we have

$$p = p(u^3), \quad q = q(u^3), \quad r = r(u^3), \quad \tilde{b}_3^{11} = \frac{p'}{2}, \quad \tilde{b}_3^{12} = q', \quad \tilde{b}_3^{13} = r',$$

which leads to

$$P_{(y)} = \begin{pmatrix} p d_y + \frac{p' u_y^3}{2} & q d_y + q' u_y^3 & r d_y + r' u_y^3 \\ q d_y & 0 & 0 \\ r d_y & 0 & 0 \end{pmatrix}.$$

In this case, (50) transform  $p, q, r$  as

$$p \rightarrow e^{-2\psi} \left( p(\varphi^3) - \frac{2(\varphi^1)' r(\varphi^3)}{(\varphi^3)'} \right), \quad q \rightarrow q(\varphi^3) - \frac{(\varphi^2)' r(\varphi^3)}{(\varphi^3)'}, \quad r \rightarrow \frac{r(\varphi^3)}{(\varphi^3)'} e^{2\psi}.$$

Thus, since  $r \neq 0$ , we can always reduce  $r$  to 1 and  $p$  and  $q$  to 0, obtaining

$$P_{(y)} = \begin{pmatrix} 0 & 0 & d_y \\ 0 & 0 & 0 \\ d_y & 0 & 0 \end{pmatrix}.$$

**Case (11)<sub>2</sub>.** Here the first structure is (11)<sub>2</sub>, and the group of transformations which preserve its form is given by

$$u^1 = e^{\psi(v^3)} v^1 + \varphi^1(v^3), \quad u^2 = e^{-\psi(v^3)} v^2, \quad u^3 = \varphi^3(v^3). \quad (52)$$

In the case where the second metric is of the form (47), conditions (5) lead to two structures, given respectively by

$$p = p(u^3), \quad q = q(u^3), \quad r = 0, \quad \tilde{b}_3^{11} = \frac{p'}{2}, \quad \tilde{b}_3^{13} = -\tilde{b}_3^{31} = \frac{q}{u^2}, \quad \tilde{b}_3^{21} = q', \quad (53)$$

and

$$\begin{aligned} p &= p(u^3) u^1 + \tilde{p}(u^3), \quad q = \kappa - \frac{p u^2}{2}, \quad r = 0, \quad \tilde{b}_1^{11} = \tilde{b}_2^{21} = \frac{p}{2}, \quad \tilde{b}_3^{11} = \frac{p' u^1 + \tilde{p}'}{2}, \quad \tilde{b}_2^{12} = -p, \\ \tilde{b}_3^{12} &= -\frac{p' u^2}{2}, \quad \tilde{b}_3^{31} = -\tilde{b}_3^{13} = \frac{\kappa}{u^2}, \end{aligned} \quad (54)$$

where  $\kappa$  is constant.

In the case where the second structure is of the form given by (48), conditions (5) lead again to two structures, where the first is the same as (53), and the second is given by

$$p = p(u^3)u^1 + \tilde{p}(u^3), \quad q = \kappa - \frac{pu^2}{2}, \quad r = r(u^3), \quad \tilde{b}_1^{11} = \tilde{b}_2^{21} = \frac{p}{2}, \quad \tilde{b}_3^{11} = \frac{p'u^1 + \tilde{p}'}{2}, \quad \tilde{b}_2^{12} = -p, \\ \tilde{b}_3^{12} = -\frac{p'u^2}{2}, \quad \tilde{b}_3^{13} = -\tilde{b}_3^{31} = r - \frac{\kappa}{u^2}, \quad \tilde{b}_2^{13} = -\tilde{b}_2^{31} = \frac{r}{u^2},$$

which corresponds to (54) replacing  $r, \tilde{b}_2^{13}, \tilde{b}_3^{13}, \tilde{b}_2^{31}, \tilde{b}_3^{31}$  with

$$r = r(u^3), \quad \tilde{b}_2^{13} = -\tilde{b}_2^{31} = \frac{r}{u^2}, \quad \tilde{b}_3^{13} = -\tilde{b}_3^{31} = r - \frac{\kappa}{u^2}.$$

Finally, considering (49), conditions (5) imply

$$p = 0, \quad q = q(u^3), \quad r = r(u^3), \quad \tilde{b}_1^{13} = -\tilde{b}_1^{31} = \frac{r}{u^2}, \quad \tilde{b}_3^{31} = -\tilde{b}_3^{13} = \frac{q}{u^2}, \quad \tilde{b}_3^{21} = q', \quad \tilde{b}_3^{23} = r'.$$

Summarising, this case leads to three different structures, namely

$$P_{(y)} = \begin{pmatrix} p \, d_y + \frac{p'u_y^3}{2} & q \, d_y & -\frac{qu_y^3}{u^2} \\ q \, d_y + q'u_y^3 & 0 & 0 \\ \frac{qu_y^3}{u^2} & 0 & 0 \end{pmatrix}, \quad P_{(y)} = \begin{pmatrix} 0 & q \, d_y & \frac{ru_y^1 - qu_y^3}{u^2} \\ q \, d_y + q'u_y^3 & 0 & r \, d_y + r'u_y^3 \\ \frac{qu_y^3 - ru_y^1}{u^2} & r \, d_y & 0 \end{pmatrix}. \quad (55) \\ P_{(y)} = \begin{pmatrix} (pu^1 + \tilde{p}) \, d_y + \frac{pu_y^1 + (p'u^1 + \tilde{p}')u_y^3}{2} & \left(\kappa - \frac{pu^2}{2}\right) \, d_y - pu_y^2 - \frac{p'u_y^2 u_y^3}{2} & r \, d_y + \frac{ru_y^2 + (r'u^2 - \kappa)u_y^3}{u^2} \\ \left(\kappa - \frac{pu^2}{2}\right) \, d_y + \frac{pu_y^2}{2} & 0 & 0 \\ r \, d_y + \frac{\kappa u_y^3 - ru_y^2}{u^2} & 0 & 0 \end{pmatrix}. \quad (56)$$

These are the more general solutions assuming the first operator given by (11)<sub>2</sub>. In these cases, a transformation of the form (52) allows to simplify these structures, but we will get more cases. Let us discuss each operator in detail.

Let us consider an operator of the form (55)<sub>1</sub>. Under change of coordinates of the form (52),  $p$  and  $q$  transform as

$$p \rightarrow e^{-2\psi} p(\varphi^3), \quad q \rightarrow q(\varphi^3).$$

Thus, if  $p$  vanishes and  $q = \xi$  is constant, we get  $\tilde{g} = \xi g$ ,  $\tilde{b} = \xi b$ , that is, a trivial operator. If  $p$  vanishes, but  $q$  is arbitrary, it can be easily reduce to  $v^3$ . Otherwise, if  $p \neq 0$ , it can be always reduced to 1, and, if  $q$  is not constant, the freedom in  $u^3 = \varphi^3(v^3)$  allows to reduce  $q$  to  $v^3$ . Therefore, in this case we get the following non-trivial canonical forms

$$P_{(y)} = \begin{pmatrix} \epsilon d_y & v^3 \, d_y & -\frac{v^3 v_y^3}{v^2} \\ v^3 \, d_y + v_y^3 & 0 & 0 \\ \frac{v^3 v_y^3}{v^2} & 0 & 0 \end{pmatrix}, \quad P_{(y)} = \begin{pmatrix} d_y & \kappa \, d_y & -\frac{\kappa v_y^3}{v^2} \\ \kappa \, d_y & 0 & 0 \\ \frac{\kappa v_y^3}{v^2} & 0 & 0 \end{pmatrix},$$

where  $\kappa$  is constant and  $\epsilon$  can be either 0 or 1.

When the operator takes the form (55)<sub>2</sub>, if  $r$  vanishes, it reduces to the first operator of the previous case with  $\epsilon = 0$ . In general, a change of coordinates given by (52) with  $\psi = \text{const}$ , transforms  $r$  and  $q$  as

$$r \rightarrow \frac{r(\varphi^3)}{(\varphi^3)', \quad q \rightarrow q(\varphi^3) - \frac{r(\varphi^3)(\varphi^1)'}{(\varphi^3)'}$$

Notice that here we have to impose the constraint  $\psi = \text{const}$ , otherwise we would have an extra function in the metric written in the new coordinates. Thus, for  $r \neq 0$ , choosing  $\varphi^1, \varphi^3$  such that  $(\varphi^3)' = r(\varphi^3)$ ,  $(\varphi^1)' = q(\varphi^3)$ ,  $r$  can be brought to 1 and  $q$  to 0, obtaining

$$P_{(y)} = \begin{pmatrix} 0 & 0 & \frac{v_y^1}{v^2} \\ 0 & 0 & d_y \\ -\frac{v_y^1}{v^2} & d_y & 0 \end{pmatrix}.$$

Finally, we have to look at the case (56). In the general case, a change of coordinates of the form (52) transforms the functions  $p, \tilde{p}$  and  $r$  as

$$p \rightarrow \left( p(\varphi^3) - \frac{2r(\varphi^3)\psi'}{(\varphi^3)'} \right) e^{-\psi}, \quad \tilde{p} \rightarrow \left( \tilde{p}(\varphi^3) + p(\varphi^3)\varphi^1 - \frac{2r(\varphi^3)(\varphi^1)'}{(\varphi^3)'} \right) e^{-2\psi}, \quad r \rightarrow \frac{r(\varphi^3)e^{-\psi}}{(\varphi^3)'}$$

Thus, if  $r \neq 0$ , we can reduce  $p$  and  $\tilde{p}$  to zero and  $r$  to 1. Otherwise, if  $r = 0$  and  $p \neq 0$ , we can brought  $p$  to 1 and  $\tilde{p}$  to zero. If both  $r$  and  $p$  are equal to zero, then  $\tilde{p}$  can be reduced to 1. Finally, if  $r = p = \tilde{p} = 0$ , we have  $\tilde{g} = \kappa g, \tilde{b} = \kappa b$ . Therefore, the canonical form of (56) can be summarised as follow

$$P_{(y)} = \begin{pmatrix} 0 & \kappa d_y & d_y + \frac{v_y^2 - \kappa v_y^3}{v^2} \\ \kappa d_y & 0 & 0 \\ d_y + \frac{\kappa v_y^3 - v_y^2}{v^2} & 0 & 0 \end{pmatrix}, \quad P_{(y)} = \begin{pmatrix} d_y & \kappa d_y & -\frac{\kappa v_y^3}{v^2} \\ \kappa d_y & 0 & 0 \\ \frac{\kappa v_y^3}{v^2} & 0 & 0 \end{pmatrix},$$

$$P_{(y)} = \begin{pmatrix} v^1 d_y + \frac{v_y^1}{2} & \left( \kappa - \frac{v^2}{2} \right) d_y - v_y^2 & -\frac{\kappa v_y^3}{v^2} \\ \left( \kappa - \frac{v^2}{2} \right) d_y + \frac{v_y^2}{2} & 0 & 0 \\ \frac{\kappa v_y^3}{v^2} & 0 & 0 \end{pmatrix}.$$

**Remark.** Allowing linear change of the independent variables  $x$  and  $y$ , one can easily see that it is always possible to reduce to zero the part of  $P_{(y)}$  which is proportional to  $P_{(x)}$ . This means that we can set  $\kappa$  equal to zero in the above canonical forms.

**Case (11)<sub>3</sub>.** Here the first structure is given by (11)<sub>3</sub>. In the first case, that is, when the second metric is given by (47), conditions (5) imply

$$p = p(u^3), \quad q = \kappa - u^3 p, \quad r = (u^3)^2 p, \quad \tilde{b}_3^{11} = \frac{p'}{2}, \quad \tilde{b}_3^{12} = -\frac{u^3 p'}{2}, \quad \tilde{b}_3^{31} = -\tilde{b}_3^{13} = \frac{2u^3 p - \kappa}{u^1 u^3 - u^2},$$

$$\tilde{b}_3^{21} = -p - \frac{u^3 p'}{2}, \quad \tilde{b}_3^{22} = u^3 p + \frac{(u^3)^2 p'}{2}, \quad \tilde{b}_3^{23} = -\tilde{b}_3^{32} = \frac{u^3(2u^3 p - \kappa)}{u^1 u^3 - u^2}.$$

Thus, the operator leads

$$P_{(y)} = \begin{pmatrix} p d_y + \frac{p' u_y^3}{2} & (\kappa - u^3 p) d_y - \frac{u^3 p' u_y^3}{2} & -\frac{(2u^3 p - \kappa) u_y^3}{u^3 u^1 - u^2} \\ (\kappa - u^3 p) d_y - \frac{(2p + u^3 p') u_y^3}{2} & (u^3)^2 p d_y + \frac{(2u^3 p + (u^3)^2 p') u_y^3}{2} & \frac{u^3(2u^3 p - \kappa) u_y^3}{u^3 u^1 - u^2} \\ \frac{(2u^3 p - \kappa) u_y^3}{u^3 u^1 - u^2} & -\frac{u^3(2u^3 p - \kappa) u_y^3}{u^3 u^1 - u^2} & 0 \end{pmatrix},$$

where  $p = p(u^3)$  and  $\kappa$  is constant. First of all, by linear change of  $x$  and  $y$ , we can set  $\kappa$  to 0. A change of coordinates which preserves  $P_{(x)}$ , namely

$$u^1 = \sqrt{\frac{v^3}{\varphi(v^3)}} v^1 + \frac{c}{\sqrt{\varphi(v^3)}}, \quad u^2 = \sqrt{\frac{\varphi(v^3)}{v^3}} v^2 + c \sqrt{\varphi(v^3)}, \quad u^3 = \varphi(v^3),$$

where  $c = \text{const}$ , transforms  $p$  into  $\frac{\varphi p(\varphi)}{v^3}$ . This means that, if  $p \neq \frac{\xi}{u^3}$  for  $\xi$  constant, we can always reduce  $p$  to 1, obtaining

$$P_{(y)} = \begin{pmatrix} d_y & -v^3 d_y & -\frac{2v^3 v_y^3}{v^3 v^1 - v^2} \\ -v^3 d_y - v_y^3 & (v^3)^2 d_y + v^3 v_y^3 & \frac{2(v^3)^2 v_y^3}{v^3 v^1 - v^2} \\ \frac{2v^3 v_y^3}{v^3 v^1 - v^2} & -\frac{2(v^3)^2 v_y^3}{v^3 v^1 - v^2} & 0 \end{pmatrix},$$

Otherwise, setting  $p = \frac{\xi}{u^3}$ , we get

$$P_{(y)} = \begin{pmatrix} \frac{\xi d_y}{u^3} - \frac{\xi u_y^3}{(u^3)^2} & -\xi d_y + \frac{\xi u_y^3}{2u^3} & -\frac{2\xi u_y^3}{u^3 u^1 - u^2} \\ -\xi d_y - \frac{\xi u_y^3}{2u^3} & \xi u^3 d_y + \frac{u_y^3}{2} & \frac{2\xi u^3 u_y^3}{u^3 u^1 - u^2} \\ \frac{2\xi u_y^3}{u^3 u^1 - u^2} & -\frac{2\xi u^3 u_y^3}{u^3 u^1 - u^2} & 0 \end{pmatrix}.$$

In the cases where the second metric is given by (48) or by (49), conditions (5) imply

$$p = r = 0, \quad q = \kappa, \quad \tilde{b}_3^{13} = -\tilde{b}_3^{31} = \frac{\kappa}{u^1 u^3 - u^2}, \quad \tilde{b}_3^{23} = -\tilde{b}_3^{32} = -\frac{u^3 \kappa}{u^1 u^3 - u^2},$$

where  $\kappa$  is constant. This leads to a trivial operator, since  $\tilde{g} = \kappa g$  and  $\tilde{b} = \kappa b$ .

## References

- [1] O.I. Bogoyavlenskij, *Invariant foliations for the Poisson brackets of hydrodynamic type*. Phys. Lett. A **360** (2007), no. 4-5, 539-544.
- [2] O.I. Bogoyavlenskij, *Tensor invariants of the Poisson brackets of hydrodynamic type*. Comm. Math. Phys. **277** (2008), no. 2, 369-384.
- [3] C.P. Boyer and J.D. Finley, *Killing vectors in self-dual Euclidean Einstein spaces*. J. Math. Phys. **23** (1982), 1126-1130
- [4] B.A. Dubrovin and S.P. Novikov, *Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method*. (Russian) Dokl. Akad. Nauk SSSR **270** (1983), no. 4, 781-785.
- [5] B.A. Dubrovin and S.P. Novikov, *Poisson brackets of hydrodynamic type*. Dokl. Akad. Nauk SSSR **279**, (1984) no. 2, 294-297.
- [6] E.V. Ferapontov and K.R. Khusnutdinova, *On the integrability of (2+1)-dimensional quasilinear systems*. Comm. Math. Phys. **248** (2004), no. 1, 187-206.
- [7] E.V. Ferapontov and K.R. Khusnutdinova, *The Haantjes tensor and double waves for multi-dimensional systems of hydrodynamic type: a necessary condition for integrability*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **462** (2006), no. 2068, 1197-1219.
- [8] E.V. Ferapontov, K.R. Khusnutdinov and S.P. Tsarev, *On a class of three-dimensional integrable Lagrangians*. Comm. Math. Phys. **261** (2006), no. 1, 225-243.
- [9] E.V. Ferapontov, P. Lorenzoni and A. Savoldi, *Hamiltonian operators of Dubrovin-Novikov type in 2D*. Lett. Math. Phys. **105** (2015), no. 3, 341-377.

- [10] E.V. Ferapontov and A.V. Odesskii, *Integrable Lagrangians and modular forms*. J. Geom. Phys. **60** (2010), no. 6-8, 896-906.
- [11] E.V. Ferapontov, A.V. Odesskii and N.M. Stoilov, *Classification of integrable two-component Hamiltonian systems of hydrodynamic type in 2+1 dimensions*. J. Math. Phys. **52** (2011), no. 7, 073505, 28 pp.
- [12] N.I. Grinberg, *On Poisson brackets of hydrodynamic type with a degenerate metric*. Russian Math. Surveys **40** (1985), no.4, 231-244.
- [13] O.I. Mokhov, *Dubrovin-Novikov type Poisson brackets (DN-brackets)*. Funct. Anal. Appl. **22** (1988), no. 4, 336-338.
- [14] O.I. Mokhov, *The classification of nonsingular multidimensional Dubrovin-Novikov brackets*. Funct. Anal. Appl. **42** (2008), no.1, 33-44.
- [15] O.I. Mokhov, *Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems*. Russian Mathematical Surveys **53** (1998), no. 3, 515-622.
- [16] A. Savoldi, *On deformations of one-dimensional Poisson structures of hydrodynamic type with degenerate metric*. arXiv:1410.3361.
- [17] A.F. Sidorov, V.P. Shapeev, and N.N. Yanenko, *The method of differential constraints and its applications in gas dynamics*, Nauka Sibirsk. Otdel., Novosibirsk, (1984), 272 pp.
- [18] I.A.B. Strachan, *Degenerate Frobenius manifolds and the bi-Hamiltonian structure of rational Lax equations*. J. Math. Phys. **40** (1999), 5058-5079.
- [19] I.A.B. Strachan, *Degenerate bi-Hamiltonian structures of hydrodynamic type*. (Russian) Teoret. Mat. Fiz. **122** (2000), no. 2, 294-304; translation in Theoret. and Math. Phys. **122** (2000), no. 2, 247-255.
- [20] S.P. Tsarev, *Geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method*. Izvestija AN USSR Math. **54** (1990), no. 5, 1048-1068.